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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2025

SECOND YEAR (BATCH 2023-27)

MATHEMATICS

Date : 10/05/2025

Time : 11.00 am – 1.00 pm

Paper : 4MTMMJC4

Full Marks : 50

[Use a separate Answer Book for each group]

[All symbols carry their usual meanings]

Group - A

Answer all. Maximum you can score is 25.

1. Let $f:[a,b]\to\mathbb{R}$ be a bounded function and P is a partition of $[a,b]$ such that $U(f,P)=L(f,P)$. Show that f is a constant function. [3]

2. A function $f:[0,1]\to\mathbb{R}$ is defined by
$$f(x)=\begin{cases}2x, & if\ \ x\ \ is\ \ rational\\ 1-x, & if\ \ x\ \ is\ \ irrational\end{cases}$$
For each $n\in\mathbb{N}$, consider the partition $P_n=\left\{0,\frac{1}{n},\frac{2}{n},.....,\frac{n-1}{n},1\right\}$ of [0,1]. Choose two suitable taggings $T_1=\{\alpha_1,\alpha_2,......,\alpha_n\}$ and $T_2=\{\beta_1,\beta_2,......,\beta_n\}$ of P_n to show that $\lim\limits_{n\to\infty}S(P_n,f,T_1)\neq\lim\limits_{n\to\infty}S(P_n,f,T_2)$. Is f integrable on [0,1]? [4]

3. A function $f:[0,1]\to\mathbb{R}$ is bounded and is discontinuous at each irrational points of $[0,1]$. Is f integrable on $[0,1]$? [3]

4. Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and $\int_a^b f=0$. Prove that there exists a point $c\in[a,b]$ such that $f(c)=0$. [3]

5. Define $f(x)=\int_x^{x+1}sin(t^2)dt$. Prove that $|f(x)|<\frac{1}{x}$, if $x>0$. [3]

6. For each $n\in\mathbb{N}$, let $f_n(x)=\lim\limits_{m\to\infty}(cos\, n!\,\pi x)^{2m}$ for all $x\in[0,1]$. Examine uniform convergence of the sequence $\{f_n\}$ on [0,1]. [4]

7. a) Show that for a function $f:[a,b]\to\mathbb{R}$, boundedness of derivative f' is a sufficient but not necessary condition for f to be of bounded variation.
   b) If a function $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then show that f is bounded. [3+3]

8. a) Suppose f is a bounded real function on $[a,b]$, and let $f^3\in R[a,b]$ (i.e. Riemann-integrable on [a,b]). Does it follow that $f\in R[a,b]$?
   b) Give an example of a function $f:[0,1]\to\mathbb{R}$, such that f is not integrable on [0,1] , but f^2 is integrable on $[0,1]$. [2+2]

Group - B

Answer all. Maximum you can score is 25.

9. a) Let $D \subseteq \mathbb{R}$ and let $\{f_n\}$ be a sequence of functions convergent pointwise on D to a function f. Let $M_n = \sup_{x \in D} |f_n(x) - f(x)|$. Prove that $\{f_n\}$ converges uniformly on D to f *iff* $\lim_{n\to\infty} M_n = 0$.

   b) Suppose $f_n(x) = \dfrac{x}{1+nx^2}, x \in [0,1]$. Examine whether $\{f_n\}$ converges uniformly on $[0,1]$.

   c) State Weierstrass' M-test on series of functions and use it to test the series $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + n^4 x^2}$. [4+5+4]

10. a) Given the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence equal to 2, find the radius of convergence of each of the following series:

    (i) $\sum_{n=0}^{\infty} a_n^k x^n$ (ii) $\sum_{n=0}^{\infty} a_n x^{kn}$ (iii) $\sum_{n=0}^{\infty} a_n x^{n^2}$

    where k is a fixed positive integer.

    b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \dfrac{2^n}{n^2} \forall n \geq 1, a_0 = 0$. [6+4]

11. a) Prove that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$ where Γ stands for the Gamma function.

    b) For $m > 0$ and $n > 0$, prove that

    $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where B stands for the Beta function. [3+4]

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