RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FOURTH SEMESTER EXAMINATION, MAY 2025

SECOND YEAR (BATCH 2023-27)

MATHEMATICS

Date : 06/05/2025

Time : 11.00 am – 1.00 pm

Paper : 4MTMMJC2

Full Marks : 50

[Use a separate Answer Book for each group]

Group - A

Answer all. Maximum you can score is 25.

1. a) Does there exist a ring epimorphism from $\mathbb{R}$ onto $\mathbb{Z}$? Justify.

   b) Is the ring $\mathbb{Q}\left[\sqrt{2}\right]$ isomorphic to the ring $\mathbb{Q}\left[\sqrt{3}\right]$? Justify your answer. [3+3]

2. Let R be a commutative ring with 1. Prove that $R[x]/\langle x\rangle$ is isomorphic to R, where $\langle x\rangle$ is the principal ideal of $R[x]$ generated by x. [3]

3. a) Are the elements $\overline{3}$ and $\overline{9}$ associates in the ring $\mathbb{Z}_{12}$? Justify your answer.

   b) In the ring $\mathbb{Z}_{10}$, prove that $\overline{5}$ is prime but not irreducible. [2+4]

4. Prove that every Euclidean domain is a principal ideal domain. [4]

5. Let p be prime and $\mathbb{Q}_p = \left\{\frac{a}{b} \in \mathbb{Q} : p \text{ does not divide } b\right\}$. Show that $\mathbb{Q}_p$ is a local ring under usual addition and multiplication. [5]

6. Let R be a commutative ring with 1. Show that R contains a maximal ideal. [6]

Group - B

Answer all. Maximum you can score is 25. All symbols carry their usual meanings.

7. Let $n = 2\times 3^2 \times 5^3 \times 7^4$

   a) Show that $\frac{\phi(n)}{n} < n^{\mu(n)} < \frac{\sigma(n)}{n}$.

   b) Express U_n as a direct product of cyclic groups of prime power order and find e(n).

   c) Show that $n \notin Q_{11}$.

   d) Show that n can be written as a sum of two squares.

   e) Apply Fermat's method of infinite discent to show that $\sqrt{n}$ is irrational.

   f) Does $\left(-\frac{1}{n}\right)$ admit a finite simple continued fraction expansion? [2+3+2+2+2+1]

8. Show that 2 is a primitive root modulo 5^k for all $k \in \mathbb{N}$. Hence solve the congruence $x^4 \equiv 7(mod\ 25)$. [3+3]

9. Find all primes p such that $6 \in Q_p$. Show that for each prime p, there exists a solution $x \in \mathbb{Z}$ of the congruence $\left(x^2-2\right)\left(x^2-3\right)\left(x^2-6\right) \equiv 0$ (mod p). [4+2]

10. If two number theoretic functions F and f are related by the formula $F(n) = \sum_{d|n} f(d)$, then prove that $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d)$. [6]

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