# RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College affiliated to University of Calcutta)

**M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2025**
**FIRST YEAR [BATCH 2024-26]**

Date : 24/05/2025

## MATHEMATICS

Time : 11.00 am – 1:00 pm

**Paper : CC 9**

Full Marks : 50

**All the symbols have their usual meanings.**

Answer **all** questions. Maximum marks you can score is 50.

1. Consider the following collections of subsets of $\mathbb{R}$.

   $\mathcal{A}_0 = \{S \mid S \text{ is an open interval in } \mathbb{R}\}$

   $\mathcal{A}_H = \{T \mid T = (a,b] \text{ for some } a < b \text{ in } \mathbb{R}\}$

   Show that $\mathcal{A}_0$ and $\mathcal{A}_H$ generate same $\sigma$-algebra. [6]

2. Let $X$ be an uncountable set. Define the collection $\mathcal{A} = \{A \subseteq X : A$ is countable or $X - A$ is countable}. Prove that $\mathcal{A}$ defines a $\sigma$-algebra on $X$. Define $\mu : \mathcal{A} \to [0,\infty)$ by $\mu(A) = 0$, if A is countable and $\mu(A) = 1$ if $X - A$ is countable. Does $\mu$ define a measure on $\mathcal{A}$? Justify. [8]

3. Consider the sequence of functions $f_n(x) = x^n + 1,\ x \in [0,1]$.

   Verify Egoroff's theorem for $\varepsilon = \frac{1}{10}$ for this sequence of functions. [5]

4. a) Let $E = \mathbb{Q} \cap [0,1]$. If $\{I_j\}_{j=1}^{n}$ are $n$-number of open intervals covering $E$, show that $\sum_{j=1}^{n} l(I_j) \geq 1$.

   b) Does the above part contradict the result that "given $\varepsilon > 0$, $\exists$ finitely many disjoint intervals $\{I_j\}_{j=1}^{n}$ such that $m(E \Delta S) < \varepsilon$ where $S = \bigcup_{j=1}^{n} I_j$ " ? [4+4]

5. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ such that $f_n(x) \leq f(x), \forall x \in \mathbb{R}$ ,for some measurable function $f$, and $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in \mathbb{R}$ . Is it true that $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$ , where $\mu$ denotes the Lebsegue measure? Justify your answer. [6]

6. Let $(X, \sigma, \mu)$ be a measure space. Show that $L^p(X)$ is a Banach space for all $1 \leq p \leq \infty$ . [15]

7. Given a bounded real valued Lebesgue measurable function $f$ on $E \subseteq \mathbb{R}$ with $m(E) < +\infty$ , show that for each $\varepsilon > 0$ there exist simple functions $\varphi_\varepsilon, \psi_\varepsilon : E \to \mathbb{R}$ such that $f(x) - \varepsilon < \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) < f(x) + \varepsilon\ \forall x \in E$ . [8]

8. Let $E \subseteq [0,1]$ be a measurable set such that $\mu(E) = 1$. Prove that $E$ is dense in [0.1]. [4]

——————— × ———————