RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. FOURTH SEMESTER EXAMINATION, MAY 2017

SECOND YEAR [BATCH 2015-17]

MATHEMATICS

Date : 02/05/2017

Time : 11 am – 1 pm

Paper : XVI [FUNCTIONAL ANALYSIS]

Full Marks : 50

Answer <u>all questions</u>. Maximum you can score 50.

[$\mathbb{K}$ in the sequel stands for $\mathbb{R}$ or $\mathbb{C}$]

1. a) Define an Inne-Product space over $\mathbb{K}$.

   b) Let $(X, <\cdot,\cdot>)$ be an inner-product space and $U : X \to X$ be a transformation (not assumed to be linear) that is onto and satisfies $< Ux, Uy > = < x, y >$ for all $x, y \in X$.

   i) Prove that U is one-to-one and that if the inverse transformation is denoted by U^{-1}, then show that $< U_x^{-1}, U_y^{-1} > = < x, y >$ and $< Ux, y > = < x, U_y^{-1} >$ for all x and y.

   ii) Prove that U is linear.

   [Hint : For (i) show Ux = Uy implies x = y by calculating $< x - y, x - y >$ in terms of U.

   For (ii) Use $< x, U_y^{-1} >$. [3+9]

2. a) Give example of a metric space X which supports a map $f : X \to X$ which is an isometry but not an onto map.

   b) When is a metric space (X,d) called topologically complete? Consider the set $\mathbb{Q}$ of rationals equipped with the usual metric $d(x, y) = | x - y |$. Is $(\mathbb{Q}, d)$ topologically complete? Give reasons.

   c) Let C [0,1] be the set of all (bounded) continuous real valued functions on [0,1]. Define the following metric on C[0,1].

   $\delta(f, g) = \int_0^1 | f - g | dx =$ The Riemann integral of the function $| f - g |$ on

   $[0,1], f, g \in C[0,1]$. Consider the following sequence $\{f_n\}$ in C[0,1]

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} \\ -2^n\left(x - \frac{1}{2}\right) + 1 & \text{if } \frac{1}{2} \le x \le \frac{1}{2} + \left(\frac{1}{2}\right)^n \\ 0 & \text{if } \frac{1}{2} + \left(\frac{1}{2}\right)^n \le x \le 1 \end{cases}$$

   Show that $\{f_n\}$ is a Cauchy sequence in the above metric space which does not converge in $\left(C[0,1], \delta\right)$. [3+4+10]

3. a) Let X be a finite-dimensional real normed space and Y be a real normed linear space. Let $T : X \to Y$ be a linear transformation. Prove that T is bounded linear transformation.

   (Hint : You can without loss of generality assume that X is $\mathbb{R}^n$ for some $n \ge 1$).

   b) Let $(X, ||\cdot||)$ be an infinite-dimensional normed linear space. Construct a one to one and onto linear transformation $T : X \to X$ which is unbounded.

   c) Let X and Y be normed spaces over the same scalar field. Suppose $T : X \to Y$ is a bounded linear transformation. Let D be a dense subset of $B(\theta, 1) = \{x \in X : || x || \le 1\}$. Show that $|| T || = \sup\{|| Tx || : x \in D\}$. ($\theta$ is the null vector of X). [4+4+6]

4. a) Use Hahn-Banach theorem to show that if $X \neq \{0\}$ is a normed linear space and x_1, x_2 are distinct point of X, then there exists a bounded linear functional f on X such that $f(x_1) \neq f(x_2)$ and $||f||=1$.

   b) Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of bounded linear maps from a Banch space X into a normed linear space Y with the condition that $\lim_{n\to\infty} f_n(x)$ exists in Y for each $x \in X$. Write $f(x) = \lim_{n\to\infty} f_n(x),\ x \in X$. Show that for any totally bounded subset X_0 of X. $\{f_n(x)\}_{n=1}^{\infty}$ converges f(x) uniformly over X_0 [You may use uniform boundedness principle] [3+4]

5. Let H be a Hilbert space over $\mathbb{C}$ and B(H) be the space of all bounded linear operators on H into H. Let $T \in B(H)$. Call T

   (i) an INVOLUTION if $T^2 = I$, the identity operator, (ii) a UNITARY operator if $TT^* = T^*T = I$.

   If $T \in B(H)$ satisfies **any two** of the following three properties (α) self adjoint (β) unitary (γ) involution, then show that T satisfies the **third** propety as well. [5]

6. Write **Yes** or **No** (no justification necessary)

   a) A reflexive Banach space is a Hilbert space.
   b) A reflexive normed linear space is a Banach space.
   c) The Dual space of separable Banach space is separable.
   d) There exist countably infinite dimensional Banach spaces.
   e) For a self-adjoint operator T on a Hilbert space, $< Tx, x >$ is real for every x in the Hilbert space. [5]

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