RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2017

FIRST YEAR [BATCH 2016-18]

MATHEMATICS

Date : 04/05/2017

Time : 11 am – 1 pm

Paper : VII [Linear Algebra II]

Full Marks : 50

[Attempt all the questions, maximum you can score 50.]

[Rings are commutative with 1 and modules are left modules]

1. Prove that $\left(\frac{\mathbb{Z}}{n\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{m\mathbb{Z}}\right) = 0$ if $\gcd(n, m) = 1$. [6]

2. An element $e \in R$ is called an idempotent if $e^2 = e$. If e is an idempotent in R, then prove that $M = eM \oplus (1-e)M$, where M is an R module. [6]

3. Let M be a finitely generated R module and $\phi : M \to R^n$ be a surjective homomorphism. Show that $\operatorname{Ker}\phi$ is a finitely generated R module. [6]

4. Prove that every free module is projective. Is the converse true? Justify your answer. [4+2]

5. Let P_1 and P_2 be R modules. Prove that $P_1 \oplus P_2$ is a projective R module if and only if both P_1 and P_2 are projective R modules. [6]

6. Let $O \to M' \to M \to M'' \to O$ be an exact sequence of R-modules. If M' and M'' are finitely generated, prove that M is also finitely generated. [6]

7. Let R be a Noetherian ring, and $\phi : R \to R$ be a ring homomorphism. Then show that if ϕ is surjective it is also injective, and hence an automorphism of R. [6]

8. Show that, for any R module M, the following statements are equivalent :
   a) M is a flat R module;
   b) M_p is a flat R_p module for each prime ideal p;
   c) M_m is a flat R_m module for each maximal ideal m. [6]

9. a) Let M be a finitely generated R-module and I be an ideal of R contained in Jacobson radical of R. If IM = M, then prove that M = O. [3]
   b) Let $x_1, x_2, ... x_n$ be elements of M whose images in M/JM generate M/JM where M is a finitely generated R-module and J is the Jacobson radical of R. Prove that $x_1, x_2, ..., x_n$ generate M. [3]

10. Suppose R is a P.I.D. Prove that M is flat R module iff M is torsion free R module. [6]

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