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M.A./M.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2017
FIRST YEAR [BATCH 2017-19]
MATHEMATICS

Date : 12/12/2017
Time : 11 am – 1 pm
Paper : I [Abstract Algebra]
Full Marks : 50

[Use a separate Answer Book for each group]

Group – A

(*Answer any two questions*) [2×10]

1. a) State the class equation for a finite group. Show that any group of order p^2, where p is prime, has a non-trivial centre. [1+2]
   b) Show that a group of order 72 is not simple. [5]
   c) Prove that the symmetric group $S_n\,(n \geq 3)$ is a semi-direct product of group A_n and $\langle (1,2) \rangle$. [2]

2. a) Let $(\mathbb{Q},+)$ be the additive group of all rational numbers. Prove that $\mathbb{Q} \not\cong \mathbb{Q} \times \mathbb{Q}$. [5]
   b) Let G be a group and N be a proper normal subgroup of G. Suppose that there does not exist any subgroup H of G such that $N \subseteq H \subseteq G$ & $N \neq H \neq G$ hold. Prove that the index of N in G is finite and equal to a prime number. [3]
   c) Let G be a non-commutative group of order p^3 where p is prime. Show that the centre of G has order p. [2]

3. a) Let G be an abelian group and let $f : G \to \mathbb{Z}$ be a surjective group homomorphism. Prove that $G \cong \operatorname{Ker} f \times \mathbb{Z}$ ($\mathbb{Z}$ denotes the set of integers) [3]
   b) Let G be a finite group and S be a non-empty set. Let G acts on S freely and transitively. Prove that $|G| = |S|$. [2]
   c) Let G be a group of order n $(n > 0, n \in \mathbb{Z})$, H be a subgroup of G and $H \neq G$. Let $|H| = m$ and $\left(n/m\right)! < 2n$. Prove that G has a proper normal subgroup different from identity. [5]

Group – B

(*Answer any three questions*) [3×10]

4. a) Let R be the ring of all real valued functions on [0,1] and $M = \left\{ f \in R : f\left(\frac{1}{2}\right) = 0 \right\}$. Prove that M is a maximal ideal of R which is prime. [3]
   b) Find all prime and maximal ideals of $\dfrac{\mathbb{R}[X]}{\langle x^2 - 3x + 2 \rangle}$, where $\mathbb{R}$ is the field of real numbers. [3]
   c) Construct a field with exactly 27 elements. [4]

5. a) Prove that in a unique factorization domain (UFD), every irreducible element is prime. [3]
   b) Deduce that for $n \geq 3$, (n is a square free integer) $\mathbb{Z}[\sqrt{n}\, i] = \{a + b\sqrt{n}\, i : a, b \in \mathbb{Z}\}$ is not a UFD. [4]
   c) Let R be a commutative ring with 1. Prove that $R[X]$ is a principal ideal domain (PID) only if R is a field. [3]

6. a) Let R be a commutative ring with 1 and f(X), g(X) $\in R[X]$, $g(X) \neq 0$ and g(X) is monic. Prove that there exist q(X), $r(X) \in R(X)$ such that $f(X) = g(X)q(X) + r(X)$ where either $r(X) = 0$ or $\deg r(X) < \deg g(X)$. [5]

b) Deduce that if $\rho : \mathbb{R}[X,Y] \to \mathbb{R}[T]$ is a ring homomorphism given by $X \mapsto T^2$ and $Y \mapsto T^3$, then $\ker \rho = (X^3 - Y^2)$. [5]

7. a) Let f be a ring epimorphism from a Noetherian ring R onto itself. Prove that f is an automorphism. [4]
   b) Prove that any Artinian domain is a field. [4]
   c) Prove that C[0,1] is neither Artinian nor Noetherian. [2]

8. Answer **<u>any five</u>** questions : [5×2]
   State true or false with proper justification.
   a) Every finitely generated ideal in a Boolean ring is principal.
   b) Every non-zero element of C[0,1] is either a unit of a zero-divisor.
   c) $\mathbb{Z}\left[\frac{1+i\sqrt{19}}{2}\right]$ is a Euclidean domain.
   d) Every Noetherian ring is Artinian.
   e) Let R be a commutative ring with 1 and I be an ideal of R such that ${}^{R}\!/_{I}$ is Noetherian. Then R is also Noetherian.
   f) Let R be a Euclidean domain with norm N and $p \in R$ be such that $N(p) = \min\{N(r) : r \in R$ is non-zero and non-unit$\}$. Then p is a prime element of R.

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