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M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2017

FIRST YEAR [BATCH 2016-18]

MATHEMATICS

Date : 19/05/2017

Time : 11 am – 1 pm

Paper : IX [Complex Analysis]

Full Marks : 50

Answer *any five* questions: [5 X 10]

1. i) Does there exist a non-constant function $f: D \to \mathbb{C}$ such that $f(z)$ and $\overline{f(z)}$ are both analytic in *D*? 3

   ii) Give an example to show that the Cauchy-Riemann equations are not sufficient for differentiability of a function $f: \mathbb{C} \to \mathbb{C}$ at a point $z \in \mathbb{C}$. 3

   iii) If $M \in S^2$ corresponds to $z \in \tilde{\mathbb{C}}$ then show that the antipodal point $-M \in S^2$ corresponds to $\left(-1/\bar{z}\right)$ (under the stereographic projection). 4

2. i) Let $f(z)$ be a non-constant entire function, then show that its image is dense in $\mathbb{C}$. 5

   ii) Let *f* be analytic on a connected open set *V*. If $[f(z)]^2 = \overline{f(z)}$ for all $z \in V$ then prove that *f* is constant on *V*. Find all possible values for such *f*. 3+2

3. i) Let $f(z)$ be a non-constant analytic function in a bounded domain *D*, continuous on the boundary ∂D and $f(z)=1$ on ∂D. Prove that there exists $z_0 \in D$ such that $f(z_0)=0$. 6

   ii) Suppose $f(z)$ is an analytic function on the closed unit disk, $f(0)=0$, and $|f(z)| \geq |e^z|$ whenever $|z|=1$. How big can $f\left(\frac{1+i}{2}\right)$ be? 4

4. i) If *C* is a closed contour around the origin, prove that $\left(\dfrac{a^n}{n!}\right) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{a^n e^{az}}{n! z^{n+1}} dz$.

   Hence deduce that $\displaystyle\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta} d\theta$. 2+3

   ii) Evaluate : $\oint_C \frac{1}{z^2} dz$, where the contour *C* is the arc of the unit circle $Im(z) \geq 0$ with the initial point –1 and the final point *i*. 5

5. i) Let *f* be an entire function which maps the real axis into itself and imaginary axis into itself. Show that $f(-z) = -f(z) \quad \forall z \in \mathbb{C}$. 4

   ii) Suppose $f: \hat{D}_R(z_0) \to \mathbb{C}$ has an essential singularity at z_0. Then $\forall r \in (0, R)$, prove that $f(\hat{D}_r(z_0))$ is dense in $\mathbb{C}$, where $\hat{D}_R(z_0)$ denotes the deleted neighbourhood of z_0. 6

6. i) State Rouche's Theorem. If $a > e$ then prove by the help of Rouche's Theorem that the equation $e^z = az^n$ has *n* roots inside the circle $|z|=1$. 2+3

   ii) Suppose $f(z)$ is entire and with a pole of order *m* at ∞. Prove that *f* is a polynomial of degree *m*. 5

7. i) Prove that $\exp\left\{\frac{1}{2}c\left(z-\frac{1}{z}\right)\right\}=\sum_{n=-\infty}^{\infty}a_n z^n$, when $z\neq 0$, where $a_n=\frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta-c\sin\theta)d\theta$. 5

ii) Find the linear fractional transformation $\phi(z)$ that maps $z_1=-2,\ z_2=-1-i,\ z_3=0$ onto the points $w_1=-1,\ w_2=0,\ w_3=1$, respectively. 5

8. i) Find the residue of the function: $f(z)=(1+z)^{-1}(1+z^2)^{-1}e^{iz}$ at $z = i$. 3

ii) Evaluate the integral $\int_0^{\infty}\frac{\cos(x)}{1+x^4}dx$. 7

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