RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2016
SECOND YEAR [BATCH 2015-17]

Date : 3/12/2016 **MATHEMATICS**

Time : 11 am – 1 pm **Paper :** XII [Ordinary Differential Equations] Full Marks : 50

Answer <u>any five</u> questions : [5×10]

1. a) Show that the initial value problem : $\frac{dy}{dx} = 3y^{2/3}$; y(0) = 0 has infinitely many solutions. [3]

   b) Consider the initial value problem $\frac{dx}{dt} = x\log x$; x(0) = a > 0. Test whether the above Initial Value Problem has any solution. If solution exists, then test whether it is unique? [4]

   c) Show that the initial value problem
   $\frac{d^2x}{dt^2} = f(t,x)$, $x(t_0) = x_0, x'(t_0) = x_1$ where f(t,x) is continuous in a domain D containing the point (t_0, x_0), is equivalent to the integral equation $x(t) = x_0 + (t-t_0)x_1 + \int_{t_0}^{t}(x-s)f(s,x(s))\,ds$. [3]

2. a) Apply Gronwall Inequality to prove the following :
   "If $x_1(t)$ and $x_2(t)$ are two differentiable functions such that $|x_1(a) - x_2(a)| \le \delta$ and $|x_i'(t) - f(t,x_i)| < \epsilon_i$ $(i = 1,2)$ for $a \le t \le b$ and if the function f(t,x) satisfies Lipschitz condition in x, then $|x_1(t) - x_2(t)| \le \delta e^{L(t-a)} + \frac{\epsilon_1 + \epsilon_2}{L}\left[e^{L(t-a)} - 1\right]$ where L is the Lipschitz constant." [6]

   b) Find the domain in which the following functions satisfy the Lipschitz condition, also find the Lipschitz constants : (i) $\frac{y}{1+x^2}$, (ii) $\frac{x}{1+y^2}$ [4]

3. a) Let x(t) be a solution of $\frac{dx}{dt} = f(t,x)$ defined on $I = \{t : a < t < b\}$, where $b < \infty$. Show that there exists a continuation of x(t) to the right if and only if $\lim_{t\to b-} x(t) = \xi$ exists and the point (b,ξ) lies in the domain of f(t,x). [6]

   b) Consider the ordinary differential equation $\frac{dx}{dt} = x^2, x(0) = 1$. Show that the largest interval of existence of the solution predicted by Peano's theorem is $\left[-\frac{1}{4} \le t < \frac{1}{4}\right]$, i.e., $x_1(t) = \frac{1}{1-x}$ in $\left[-\frac{1}{4} \le t \le \frac{1}{4}\right]$. Prove that $x_2(t) = \frac{1}{1-x}$, if $\frac{1}{4} \le t \le \frac{7}{16}$. [4]

4. a) If one solution u(t) of the homogeneous differential equation
   $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x = 0,\ p,q \in C[a,b]$ is known, prove that a second linearly independent solution v(t) is given by $v(t) = Au(t) + u(t)\int_a^t \frac{W(a)e^{-\int_a^s p(\xi)d\xi}}{u^2(s)}ds$,
   where A is an arbitrary constant and W(a) is the Wronskian when t = a. [4]

   b) Let W(x) be the Wronskian of two linearly independent solutions of the linear ODE $a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$. Prove that : $W'(x) = -\frac{a_1(x)}{a_0(x)}W(x)$. [3]

c) Let W(x) be the Wronskian of two linearly independent solutions of the linear ODE $\frac{d^2y}{dx^2}+(1+x^2)\frac{dy}{dx}+P(x)y=0$, where P(x) is a continuous function for x > 0. Let W(1) = a, W(2) = b & W(3) = c. Use 4(b) to prove : $\frac{a}{|a|}=\frac{b}{|b|}=\frac{c}{|c|}$. [3]

5. a) Let u and v be nontrivial solutions of $\frac{d^2x}{dt^2}+p(t)x=0$, $\frac{d^2x}{dt^2}+q(t)x=0$ respectively, on an interval I, where $p(t)\neq q(t)$ and $p(t)\geq q(t)$. Show that between any two zeros of v there is at least one zero of u. [5]

Hence deduce that if $q(t)\leq 0$, then no trivial solution of the differential equation $\frac{d^2x}{dt^2}+q(t)x=0$, can have more than one zero. [2]

b) Let p(t) > 0 and let p, q be continuous and p be differentiable on [a,b]. Show that the only solution of $L[x]=\frac{d}{dt}\left(p(t)\frac{dx}{dt}\right)+q(t)x=0$

which has an infinite number of zeros on [a,b] is the trivial solution. [3]

6. a) Let the coefficient p, q and s in the regular Sturm-Liouville system be continuous $\frac{d}{dt}\left(p\frac{dx}{dt}\right)+(q+\lambda s)x=0$, $A_1x(a)+A_2x'(a)=0$, $B_1x(b)+B_2x'(b)=0$

where the constants A_1 and A_2, likewise B_1 and B_2, are not both zero.

Show that the eigen functions φ_j and φ_k corresponding to λ_j and λ_k are orthogonal with respect to the weight function *s* in [a,b]. [5]

b) If $\varphi_1(t)$ and $\varphi_2(t)$ are any two solutions of $\frac{d}{dt}\left(p\frac{dx}{dt}\right)+(q+\lambda s)x=0$, on [a, b].

Show that p(t) W $(\varphi_1,\varphi_2;t)$ = constant, where W is the Wronskian. [3]

Hence deduce that an eigen function of a regular Sturm-Liouville system is unique except for a constant factor. [2]

7. a) Consider the system of ODE : $\frac{dx}{dt}=Ax$; $x(0)=\begin{pmatrix}1\\1\\1\end{pmatrix}$ where $x(t)=\begin{pmatrix}X_1(t)\\X_2(t)\\X_3(t)\end{pmatrix}$, $A=\begin{pmatrix}-2&1&0\\0&-2&1\\0&0&-2\end{pmatrix}$.

Show that if the solution of the above ODE is x(t), then $\lim_{t\to 0}\|x(t)\|=0$ where $\|x(t)\|=\sqrt{(X_1(t))^2+(X_2(t))^2+(X_3(t))^2}$. [6]

b) Determine the fundamental matrix of the system $\begin{pmatrix}\dot{x}_1\\\dot{x}_2\end{pmatrix}=\begin{pmatrix}2&2\\1&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$. [4]

8. Consider the linear autonomous system with constant coefficient $\dot{x}=Ax$ .... (1)

where A is a nonsingular real 2×2 matrix.

Determine the equilibrium point of the system (1). Investigate the nature of critical points of the system (1). [10]

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