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(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2016

SECOND YEAR [BATCH 2015-17]

MATHEMATICS

Date : 5/12/2016

Time : 11 am – 1 pm

Paper : XIII [Mathematical Methods]

Full Marks : 50

[Use a separate Answer Book for each Group]

Group – A

(***Answer any two questions***) [2×17]

All notations and symbols have their usual meanings :

1. a) What are test functions? How convergence in $\mathcal{D}$ is defined? Give definition of generalized functions. What are regular generalized functions?

   b) How Dirac delta function is defined. Show that this is a singular generalized function.

2. a) Show that

   $$f_{\in}(x) = \frac{1}{\pi}\frac{\in}{x^2+\in^2} \to \delta(x)$$

   as $\in \to 0$ in $\mathcal{D}'(R')$.

   b) Find the eigen values and eigen functions of the integral equation,

   $$\varphi(x) = \lambda\int_0^{\pi}\cos(x+t)\varphi(t)dt$$

3. a) Find the resolvent kernel of the integral equation,

   $$\varphi(x) = \lambda\int_0^1 xe^t\varphi(t)dt + x$$

   and hence find its solution.

   b) Use Green's function to solve the boundary-value problem

   $$\frac{d^2y}{dx^2} + y = x^2,\ 0 < x < \frac{\pi}{2},\ y(0) = y\left(\frac{\pi}{2}\right) = 0.$$

4. a) Find the derivative of the generalized function generated by the step function,

   $$\theta(x) = 1,\ x > 0$$
   $$= 0,\ x < 0$$

   b) Find the solution of the integral equation,

   $$\frac{1}{\sqrt{\pi}}\int_0^x \frac{\varphi(t)}{\sqrt{x-t}}dt = f(x).$$

Group – B

(***Answer any one question***) [1×16]

5. a) If the functions f and g possess piecewise continuous derivatives in $(-\infty,\infty)$ and they are absolutely integrable in $(-\infty,\infty)$ and also let f, g, g' be square integrable in the same domain then prove that $\int\limits_{-\infty}^{\infty} F(s)G(s)e^{-ist}ds = \int\limits_{-\infty}^{\infty} f(u)g(t-u)du$

   where F(s) and G(s) are the Fourier transforms of f and g respectively. [5]

   b) State and prove the Modulation theorem for Fourier transform. [3]

c) Use Fourier transform method to determine the displacement component y(x,t) of an infinite string satisfying the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, given that the string is initially at rest and that the initial displacement is f(x), $-\infty < x < \infty$. Show that the solution can also be put in the form $y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$. [8]

6. a) If f(t) exists finitely and continuous at each point of $(-\infty, \infty)$ then prove that $\lim_{|t|\to\infty} f(t) = 0$. Using this result show that

$F\{f'(t);s\} = -is\,F\{f(t);s\}$

provided $f'(t)$ exists finitely in $(-\infty, \infty)$ [5+2]

b) Find the finite Fourier sine and cosine transform of the function $f(x) = \left(1 - \frac{x}{\pi}\right)^2$. [3]

c) Using Fourier transform, solve the partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the half-plane $H = \{(x, y) : -\infty < x < \infty$ and y>0} subject to the conditions—

i) u(x,0) = f(x) for $-\infty < x < \infty$, and

ii) $u(x, y) \to 0$ as $(x^2 + y^2)^{1/2} \to \infty$. [6]

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