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M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2016

FIRST YEAR [BATCH 2015-17]

MATHEMATICS

Date : 25/5/2016

Time : 11 am – 1 pm

Paper : IX

Full Marks : 50

Answer any five questions : [5×10]

1. a) Define Winding number of a smooth closed curve with respect to a point not on the curve. If $\gamma:[0,1]\to\mathbb{C}$ is a smooth closed curve and $\alpha\notin\gamma$ then prove that the Winding number $n(\gamma;\alpha)$ of γ with respect to α is an integer. [1+3]

   b) State Morerea's theorem. Use it to prove the following : Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function f(z) in a region Ω, uniformly on every compact subset of Ω. Then prove that f(z) is analytic in Ω. Moreover, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω. [1+4+1]

2. a) If $f(z)=a_0+a_1z+...+a_nz^n$ is a polynomial of degree $n\geq 1$, then show that the equation $f(z)=0$ has at least one root in $\mathbb{C}$. [4]

   b) If P be a non-constant polynomial in $\mathbb{C}$. Prove that $P(\mathbb{C})=\mathbb{C}$. Is the result true for any entire function? Justify your answer. [3]

   c) Prove that a non-constant entire function cannot leave a non-empty open disc from its image set, however small its radius is. [3]

3. a) Let $\gamma:|z-z_0|=R$ be the circle of convergence of the series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$. Show that the series is uniformly convergent on every compact subset of $\text{int}\,\gamma$. [4]

   b) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^z}$ is uniformly convergent in the closed region $\text{Re}\,z\geq 1+\delta, \delta>0$. [3]

   c) Extend the series $\sum_{n=1}^{\infty}\frac{1}{n^z}$ analytically (having only one pole at z = 1) to the region Re z > 0. [3]

4. a) i) Prove that given any $n(\geq 1)$ points on the unit circle of the complex plane there is always a point on the same circle such that the product of the distance of the given point to this point is at least one. [3]

   ii) Is it possible to partition $\mathbb{N}$, the set of all natural numbers, into finite union of arithmetic progression with distinct common differences. [3]

   b) Let f be an entire function such that $|f(z)|\leq A|z^p|$, $\forall z\in\mathbb{C}$ and for some $p\geq 0$, A being a positive constant. Prove that f is a polynomial of degree not exceeding p. [4]

5. a) Show that the Laurent expansion of $f(z)=\exp\left(\frac{u}{2}\left(z-\frac{1}{z}\right)\right), u\in\mathbb{C}$ in the region $0<|z|<\infty$ is given by $f(z)=\sum_{n=-\infty}^{n=\infty}a_nz^n$ where $a_n=\frac{1}{\pi}\int_0^{2\pi}\cos(n\theta-u\sin\theta)d\theta$ and $a_{-n}=(-1)^n$, n = 0,1,2,... [6]

   b) Prove that an analytic function comes arbitrarily close to any complex value in every neighbourhood of its essential singularity. [4]

6. a) State Rouche's theorem on analytic function. Use it to find the number of roots of the polynomial equation $z^7-2z^5+6z^3-z+1=0$ have in the disc $|z|<1$. [1+3]

b) Let f be analytic within and on a simple closed curve γ except for a finite number of poles within γ and $f(z) \neq 0$ on γ. Also, let φ be analytic within and on γ. If $\alpha_1,...,\alpha_m$ be the zeros of f within γ of respective orders $p_1,...,p_m$ and $\beta_1,...,\beta_n$ be the poles of f within γ of respective orders $q_1,...q_n$, show that $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\varphi(z)dz = \sum_{k=1}^{m} p_k \varphi(\alpha_k) - \sum_{k=1}^{n} q_k \varphi(\beta_k)$. [6]

7. a) Evaluate **<u>any one</u>** of the following real integrals by contour integration : [1×5]

i) $\int_0^\infty \frac{\sin x}{x}dx$

ii) $\int_0^\infty \frac{x^2 dx}{x^4 + 5x^2 + 6}$

b) Prove that the sum of the residues of the function $f(z) = \frac{e^{-z}}{z^2 - a^2}$ at its poles is $-\frac{1}{a}\sinh a$. [2]

c) Find the residues of the function $f(z) = \sec\frac{1}{z}$ at its poles. [3]

8. a) Prove that any bilinear transformation can be expressed as a composition of translation, dilation and inversion (some of these may be missing). [5]

b) Show that the bilinear transformation which carries the points z = i, 0, –i into the points w = 0, –1, ∞ maps

i) the real axis $\text{Im} z = 0$ on the circle $|w| = 1$,

ii) the upper half-plane $\text{Im} z > 0$ on the disk $|w| < 1$

iii) lower half-plane $\text{Im} z < 0$ on the region $|w| > 1$. [5]

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