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(Residential Autonomous College affiliated to University of Calcutta)

**M.A./M.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2016**

**FIRST YEAR [BATCH 2016-18]**

**MATHEMATICS**

Date : 14/12/2016

Time : 11 am – 1 pm

**Paper : I**

Full Marks : 50

[ $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{Z}$ denotes the set of integers]

## Group – A

(***Answer any two questions***) [2×11]

1. Let $R = \dfrac{\mathbb{Q}[X]}{(X^3 - X)}$.
   a) Prove that there exists an isomorphism $\phi$ from $\mathbb{R}$ to the product ring $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$. [3]
   b) For which element e of $R, \phi(e) = (1,0,0)$? [3]
   c) Describe all prime an maximal ideals of R. Does R contain any idempotent? [5]

2. a) Let R, S be two commutative ring with unity and $\theta : R \to S$ be a ring homomorphism. Prove that the inverse image of a prime ideal of S is a prime ideal of R. Does the result hold true if we replace prime ideal by maximal ideal? [4]
   b) Let R be a commutative ring with identity. Prove that every maximal idea of R is prime. Does the converse hold? Justify your answer. [3+1]
   c) Deduce that if in a commutative ring with unity the set of all non-unit elements forms a maximal ideal, then the ring has only one maximal ideal. [3]

3. a) Let R be a Principal Ideal Domain. Show that a non-zero proper ideal I of R is a prime ideal of R if and only if I is generated by some prime element of R. [4]
   b) Let R be a commutative ring with unity such that every nonzero prime ideal of R is a maximal ideal. Can you conclude that R is a Principal Ideal Domain? [3]
   c) If R is a commutative ring with unity such that the polynomial ring R[X] is a Principal Ideal Domain, then prove that R is a field. [4]

4. a) Prove that the ring of real valued continuous functions of [0,1] is not Notherian, Is it Artinian? [3]
   b) Prove that the ring $\dfrac{\mathbb{R}[X,Y]}{(X^2+Y^2)}$ is a Notherian integral domain. [4]
   c) Prove that an Artinian domain is a field. [4]

## Group – B

(***Answer any one question***) [1×14]

5. a) Show that $\mathbb{R}/\mathbb{Z}$ is isomorphic to $S^1$, where $S^1$ denotes the multiplicative group of complex numbers z of unit modulus and $\mathbb{R}$ denotes the additive group of reals. Does there exist any subgroup G of $\mathbb{R}$ such that $\mathbb{R}/G \simeq \mathbb{Z}$? [5]
   b) Prove that the group $\mathbb{Q}$ has no proper subgroups of finite index. Deduce that $\mathbb{Q}/\mathbb{Z}$ has no subgroups of finite index. [4]
   c) Let H and K be two subgroups of G such that one of them is contained in the normalizer of other in G. Prove that HK = KH. [5]

6. a) If G is a finite group of order n and p is the smallest prime dividing n, then prove that any subgroup of index p is normal. [4]

b) Let G be a non-commutative group of order $p^3$ where p is prime. Show that centre of G contains exactly p elements. [4]

c) Show that a group G of order 30 cannot be simple. Moreover, show that G has a subgroup of index 2. [6]

## **Group – C**

7. State whether the following statements are TRUE or FALSE with proper justification. Anwer **any four** : [4×3·5]

a) If $H \subseteq K$ are subgroups of an Abelian group G such that $\frac{G}{H} \cong \frac{G}{K}$, then H = K.

b) The additive group of rational numbers is isomorphic to the multiplicative group of positive rational numbers.

c) Any finitely generated subgroup of $\mathbb{Q}$ is cyclic.

d) The ring of all real valued continuous functions on [0,1] is an integral domain.

e) Any nonzero homomorphism from a field to itself is an isomorphism.

f) There exists $f(X) \in \mathbb{R}[X]$ of degree greater than 2 such that $\frac{R[X]}{(f(X))}$ is an integral domain.

g) In a Boolean ring ($a^2 = a$, for every element of the ring) every finitely generated ideal is principal.

h) $\mathbb{Z}[x]$ is a Principal Ideal Domain.

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